Unique positive solution for alternative discrete Painlevé I equation

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Joint work with P. Clarkson and W. Van Assche

A Monic Orthogonal Polynomial Sequence (MOPS) $\{P_n\}_{n\geqslant 0}$ is defined by

$$\langle u_0, P_n P_k \rangle = N_n \delta_{n,k} \ , \ \text{with} \ \ N_n \neq 0.$$

where u_0 is the first element of the corresponding dual sequence (canonical form).

- It always satisfies the second order recurrence relation

$$P_{n+1}(x) = (x - \beta_n) P_n(x) - \gamma_n P_{n-1}(x)$$

with $P_0 = 1$ and $P_{-1} = 0$ and

$$\beta_n = \frac{\langle u_0, xP_n^2 \rangle}{\langle u_0, P_n^2 \rangle} \qquad \text{and} \qquad \gamma_{n+1} = \frac{\langle u_0, P_{n+1}^2 \rangle}{\langle u_0, P_n^2 \rangle} \neq 0, \ n \in \mathbb{N}$$

in this case...

the Hankel determinant

$$\Delta_n(u_0) = \det\left[(u_0)_{i+j}\right]_{0\leqslant i,j\leqslant n} \neq 0\ ,\ \ n \geqslant 0, \quad \text{with } (u_0)_k = < u_0, x^k >,$$

and

$$P_n(x) = \frac{1}{\Delta_{n-1}(u_0)} \begin{vmatrix} 1 & (u_0)_1 & \dots & (u_0)_{n-1} & (u_0)_n \\ (u_0)_1 & (u_0)_2 & \dots & (u_0)_n & (u_0)_{n+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ (u_0)_{n-1} & (u_0)_n & \dots & (u_0)_{2n-2} & (u_0)_{2n-1} \\ 1 & x & \dots & x^{n-1} & x^n \end{vmatrix} ,\ \ n \geqslant 0,$$

Notice that the recurrence coefficients β_n and γ_{n+1} are also given by

$$\beta_n = \frac{\Delta^*_{n+1}(u_0)}{\Delta_n(u_0)} - \frac{\Delta^*_n(u_0)}{\Delta_{n-1}(u_0)} \quad , \quad \gamma_{n+1} = \frac{\Delta_{n-1}(u_0)\Delta_{n+1}(u_0)}{\Delta_n(u_0)^2}$$

where

Classical polynomials

Definition. A MOPS $\{P_n\}_{n\geqslant 0}$ is classical iff $\{\frac{1}{n+1}P'_{n+1}(x)\}_{n\geqslant 0}$ is also a MOPS.

The classical polynomials $\{P_n\}_{n\geqslant 0}$ orthogonal for u_0 are characterised by

(a) There exists a pair of polynomials (ϕ, ψ) such that

$$D(\phi u_0) + \psi u_0 = 0$$

where $\max(\deg \phi - 2, \deg \psi - 1) = 0$ with ϕ monic.

(b) There exists a pair of polynomials (ϕ, ψ) and a sequence $\{\lambda_n \neq 0\}_{n\geqslant 0}$ such that

$$\phi(x)P''_{n+1}(x) - \psi(x)P'_{n+1}(x) = \lambda_n P_{n+1}(x) \ , \ \ n \geqslant 0$$

(c) There exists a monic polynomial ϕ with $\deg \phi \leqslant 2$ such that

$$\Phi(x)P'_{n+1}(x) = \sum_{\nu=n}^{n+\deg \Phi} \theta_{n,\nu} P_\nu(x) \qquad \text{with} \qquad \theta_{n,n}\theta_{n,n+t} \neq 0, \quad n \geqslant 0.$$

(d) There exists a monic polynomial ϕ with $\deg \phi \leqslant 2$ and a sequence $\{\theta_n \neq 0\}_{n\geqslant 0}$ such that $\quad P_n u_0 = \theta_n D^n\left(\phi^n u_0\right) \ , \ \ n \geqslant 0.$

A MOPS $\{P_n\}_{n\geqslant 0}$ for u_0 is semiclassical when

$$D(\phi u_0) + \psi u_0 = 0$$

where $\max(\deg\phi - 2, \deg\psi - 1) = s \geqslant 0$. (We assume ϕ monic). In this case...

(a) $\phi(x)P''_{n+1}(x) - \psi(x)P'_{n+1}(x) = q_s(x;n)P_{n+1}(x) + \sum_{\nu=n-s+1}^{n} \lambda_{n,\nu}P_\nu(x),\ n \geqslant s.$

(b) $\Phi(x)P'_{n+1}(x) = \sum_{\nu=n-s}^{n+\deg\Phi} \theta_{n,\nu}P_\nu(x)$ with $\theta_{n,n-s}\theta_{n,n+t} \neq 0, \quad n \geqslant 0.$

(c) $\Phi(x)P'_{n+1}(x) = A_n(x)P_{n+1}(x) - B_n(x)P_n(x)$ with $\deg A_n \leq s+1$ and $\deg B_n \leq s$

(d) $J(x;n)P''_{n+1}(x) - K(x;n)P'_{n+1}(x) + L(x;n)P_{n+1}(x) = 0,\ n \geqslant 0.$

with $\deg J(x;n) \leq 2s+2$, $\deg K(x;n) \leq 2s+1$ and $\deg J(x;n) \leq 2s$

multiple orthogonal polynomials

Consider the monic polynomials $P_{n,m}$ of degree $n+m$ for which

$$\int_{\Gamma_0 \cup \Gamma_1} x^j P_{n,m}(x) \exp\left(-x^3 + tx\right) dx = 0, \; j = 0, \ldots, n-1,$$
$$\int_{\Gamma_0 \cup \Gamma_2} x^j P_{n,m}(x) \exp\left(-x^3 + tx\right) dx = 0, \; j = 0, \ldots, m-1,$$

with $\Gamma_k = \{z \in \mathbb{C} : \arg z = e^{2k\pi i/3}\}, \; k = 0, 1, 2.$

Rodrigues' formula:

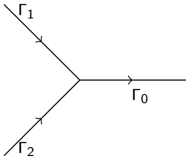
$$\mathrm{e}^{-x^3+tx} P_{n,n+m}(x) = \frac{(-1)^n}{3^n} \frac{d^n}{dx^n} \left(\mathrm{e}^{-x^3+tx} P_{0,m}(x)\right)$$
$$\mathrm{e}^{-x^3+tx} P_{n+m,n}(x) = \frac{(-1)^n}{3^n} \frac{d^n}{dx^n} \left(\mathrm{e}^{-x^3+tx} P_{m,0}(x)\right)$$

where $P_{m,0}$ and $P_{0,m}$ are orthogonal polynomials...

and $P_{k,k}$ are d-orthogonal polynomials

Orthogonality relation

$$\int_{\Gamma_0 \cup \Gamma_1} x^j P_{n,0}(x) \exp\left(-x^3 + tx\right) dx = 0, \ j = 0, \ldots, n-1,$$



Observe that

$$W'(x) + \left(3x^2 - t\right) W(x) = 0,$$

with $\quad W(x) = \exp\left(-x^3 + tx\right)$

orthogonal polynomials

Orthogonality relation

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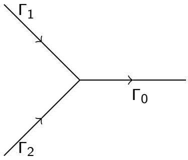

Observe that

$$W'(x) + \left(3x^2 - t\right) W(x) = 0,$$

with $W(x) = \exp\left(-x^3 + tx\right)$

Recurrence relation

$$xP_{n,0}(x) = P_{n+1,0}(x) + \hat{b}_n \, \mathrm{e}^{i\pi/3} P_{n,0}(x) + \hat{a}_n \, \mathrm{e}^{-i\pi/3} P_{n-1,0}(x),$$

where

$$\hat{a}_n + \hat{a}_{n+1} = \hat{b}_n^2 - \frac{t}{3},$$

$$3 \; \hat{a}_n \left(\hat{b}_n + \hat{b}_{n-1}\right) = n,$$

with $\hat{a}_0 = 0$ and $\hat{b}_0 = \frac{Ai'(3^{-1/3}t)}{Ai(3^{-1/3}t)}$.

about the recurrence coefficients

We investigate the system of two nonlinear equations

$$a_n + a_{n+1} = b_n^2 - t, \tag{1a}$$

$$a_n(b_n + b_{n-1}) = n, \tag{1b}$$

which is known as an **alternative discrete Painlevé I equation (alt d-$\mathrm{P_I}$)**.

These equations arise, for example, when one wants to

- compute the recurrence coefficients of multiple orthogonal polynomials with an exponential cubic weight (Filipuk, Van Assche & Zhang, 2014).
- find the recurrence coefficient of orthogonal polynomials with an exponential cubic weight (Bleher & Deaño, 2013) and (Magnus, 1995).

Uniqueness of the positive solution

Theorem (Clarkson, AL & Van Assche'16)

For $t \geq 0$, there exists a unique solution of

$$a_n + a_{n+1} = b_n^2 - t, \tag{2a}$$

$$a_n(b_n + b_{n-1}) = n, \tag{2b}$$

with $a_0(t) = 0$ for which $a_{n+1}(t) > 0$ and $b_n(t) > 0$ for all $n \geq 0$. This solution corresponds to the initial value

$$b_0(t) = -\frac{\mathrm{Ai}'(t)}{\mathrm{Ai}(t)}.$$

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Theorem (Clarkson, AL & Van Assche'16)

For $t \geq 0$, there exists a unique solution of

$$\frac{n+1}{b_n + b_{n+1}} + \frac{n}{b_n + b_{n-1}} = b_n^2 - t. \tag{3}$$

for which $b_n(t) \geq 0$ for all $n \geq 0$. This solution corresponds to the initial value

$$b_0(t) = -\frac{\mathrm{Ai}'(t)}{\mathrm{Ai}(t)}.$$

Observe that the value of b_1 is fixed once b_0 is known, in particular one has

$$b_1(t) = -b_0(t) + \frac{1}{b_0(t)^2 - t}$$

Hence the solutions of

$$\frac{n+1}{b_n + b_{n+1}} + \frac{n}{b_n + b_{n-1}} = b_n^2 - t,$$

only depend on one free parameter $b_0(t)$. This is reflected by $a_0(t) = 0$.

Proof of the uniqueness

A positive solution $(b_n)_{n\geq 0}$ implies that $a_0 = 0$ and $a_n > 0$ for $n \geq 1$.

Let $T : \mathbb{R}_+^{\mathbb{N}} \longrightarrow \mathbb{R}_+^{\mathbb{N}}$ such that $((Tx)_n)_{n\geq 0}$ is implicitly defined by

$$\frac{n+1}{(Tx)_n + x_{n+1}} + \frac{n}{(Tx)_n + x_{n-1}} = (Tx)_n^2 - t, \qquad n \geq 1 \tag{4}$$

and

$$\frac{1}{(Tx)_0 + x_1} = (Tx)_0^2 - t. \tag{5}$$

Hence, $(Tx)_n$ is a solution of

$$\frac{n+1}{y + x_{n+1}} + \frac{n}{y + x_{n-1}} = y^2 - t. \tag{6}$$

Idea: to show that T is a contraction on a complete set in this space.

Proof of the uniqueness (cont.)

Since $x_{n+1} > 0$ and $x_{n-1} > 0$, it follows from (4)

$$x_n = (Tx)_n = t + \sqrt{\frac{n+1}{(Tx)_n + x_{n+1}} + \frac{n}{(Tx)_n + x_{n-1}}} \;\leq\; t + \sqrt{\frac{2n+1}{(Tx)_n}}$$

and therefore

$$(Tx)_n(((Tx)_n)^2 - t) \leq 2n+1.$$

There is only one positive root of the equation

$$y(y^2 - t) = 2n+1$$

which is

$$B_n(t) = \frac{1}{2^{1/3}}\left(\left(2n+1-\sqrt{(2n+1)^2 - \frac{4}{27}t^3}\right)^{1/3} + \left(2n+1+\sqrt{(2n+1)^2 - \frac{4}{27}t^3}\right)^{1/3}\right)$$

Hence,

$$0 \leq (Tx)_n \leq B_n(t),\ \ n \geq 0.$$

Proof of the uniqueness (cont.)

Since $(Tx)_n \leq B_n(t)$ and $\frac{n+1}{(Tx)_n+x_{n+1}} + \frac{n}{(Tx)_n+x_{n-1}} = (Tx)_n^2 - t$, then

$$(Tx)_n^2 - t \geq \frac{n+1}{(Tx)_n + B_{n+1}(t)} + \frac{n}{(Tx)_n + B_{n-1}(t)} \quad \text{for} \quad n \geq 0. \qquad (7)$$

Proof of the uniqueness (cont.)

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Observe that for $z \in [0, \infty)$ the function $f(z) = \dfrac{1}{(y + R(z,t))}$ with

$$R(z,t) = z^{1/3} \left\{ \left(1 - \sqrt{1 - \left(\frac{t}{3z^{2/3}}\right)^3}\right)^{1/3} + \left(1 + \sqrt{1 - \left(\frac{t}{3z^{2/3}}\right)^3}\right)^{1/3} \right\}$$

is convex on $(0, \infty)$, so that

$$\lambda f(z_1) + (1-\lambda) f(z_2) \geq f(\lambda z_1 + (1-\lambda) z_2), \qquad \lambda \in [0,1],$$

whenever $0 < z_1 \leq z_2 < \infty$.
With $z_1 = n + 3/2$, $z_2 = n - 1/2$ and $\lambda = 1/2$, then this gives for $y = (Tx)_n$

$$\frac{1}{(Tx)_n + R\left(n + \frac{3}{2}, t\right)} + \frac{1}{(Tx)_n + R\left(n - \frac{1}{2}, t\right)} \geq \frac{2}{(Tx)_n + R\left(n + \frac{1}{2}, t\right)} \quad \text{for} \quad n \geq 1.$$

Proof of the uniqueness (cont.)

Since $(Tx)_n \le B_n(t)$ and $\frac{n+1}{(Tx)_n + x_{n+1}} + \frac{n}{(Tx)_n + x_{n-1}} = (Tx)_n^2 - t$, then

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With $z_1 = n + 3/2$, $z_2 = n - 1/2$ and $\lambda = 1/2$, then this gives for $y = (Tx)_n$

$$\frac{1}{(Tx)_n + R\left(n + \frac{3}{2}, t\right)} + \frac{1}{(Tx)_n + R\left(n - \frac{1}{2}, t\right)} \ge \frac{2}{(Tx)_n + B_n(t)} \quad \text{for} \quad n \ge 1.$$

Proof of the uniqueness (cont.)

Hence

$$((Tx)_n^2 - t)\,((Tx)_n + B_n(t)) \geq 2n, \quad \text{for} \quad n \geq 1.$$

With $c = \dfrac{(Tx)_n}{B_n(t)}$, it follows

$$(c+1)\Big((c^2-1)B_n(t)^3 + (2n+1)\Big) \geq 2n, \;\; n \geq 1,$$

because $tB_n(t) = B_n(t)^3 - (2n+1)$. Consequently, $2c^2(c+1) \;\geq\; \frac{4n}{2n+1} \geq \frac{4}{3}$

therefore

$$(Tx)_n \geq c_1 B_n(t), \;\; \text{for} \;\; n \geq 1, \;\; \text{with} \;\; c_1 = 0.6379714.$$

Besides...

$$(Tx)_0 \;\geq\; 0.47533 \; B_1(t) \geq 0.47533 \cdot 3^{1/3} = 0.685544.$$

Proof of the uniqueness (cont.)

With

$$0.685544 \ \leq (Tx)_0 \leq B_0(t) \quad \text{and} \quad 0.63797 \ B_n(t) \ \leq \ (Tx)_n \ \leq \ B_n(t),$$

we can deduce

$$\frac{|(Tx)_0 - (Ty)_0|}{B_0(t)} \leq c_2 \frac{|x_1 - y_1|}{B_1(t)}, \qquad c_2 = 0.568967\ldots$$

and

$$\frac{|(Tx)_n - (Ty)_n|}{B_n(t)} \leq c_3 \ \sup_{n\geq 0} \frac{|x_n - y_n|}{B_n(t)}, \qquad c_3 = 0.928273\ldots.$$

Proof of the uniqueness (cont.)

Hence, with the norm

$$\|x\| = \sup_{n\geq 0} \frac{|x_n|}{B_n(t)}$$

this gives

$$\|Tx - Ty\| \leq 0.928273\|x - y\|$$

which shows that T is a contraction.

Since the unit ball with the norm $\|\cdot\|$ is a complete metric space, one can use Banach's fixed point theorem to conclude that T has a unique fixed point b for which $Tb = b$.

The sequence $b = (b_n)_{n\geq 0}$ is positive and it is a solution of the alternative discrete Painlevé equation

$$\frac{n+1}{b_n + b_{n+1}} + \frac{n}{b_n + b_{n-1}} = b_n^2 - t.$$

□

Remarks about the behaviour of b_n

For any $t \geq 0$, we have

- $\sqrt{t} \; < \; b_n(t) \; < \; B_n(t)$
- $\lim_{t\to+\infty} \left(a_n(t)\, \frac{2\sqrt{t}}{n} \right) = 1$ $\qquad$ $\lim_{t\to\infty} \frac{b_n(t)}{\sqrt{t}} = 1$
- $\lim_{n\to\infty} \frac{a_n(t)}{n^{2/3}} = \frac{1}{2}$, $\qquad$ $\lim_{n\to\infty} \frac{b_n(t)}{n^{1/3}} = 1.$

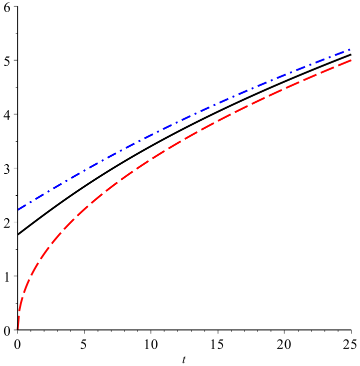
  In fact, as $t \to \infty$,

  $$b_n(t) = \sqrt{t} + \frac{2n+1}{4t} - \frac{12n^2+12n+5}{32\, t^{5/2}} + \frac{(2n+1)(16n^2+16n+15)}{64t^4} + \mathcal{O}(t^{-11/2})$$

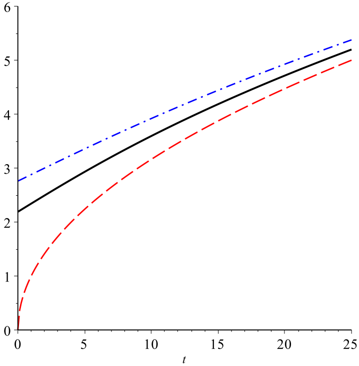
- The upper bound $B_n(t)$ is a concave increasing function of both n and t, where

  $$B_n(t) = (n+\tfrac{1}{2})^{1/3} \left\{ \left[1 - \sqrt{1 - \frac{t^3}{27(n+\frac{1}{2})^2}} \right]^{1/3} + \left[1 + \sqrt{1 - \frac{t^3}{27(n+\frac{1}{2})^2}} \right]^{1/3} \right\}$$

Comparison between $b_n(t)$, and the corresponding upper bounds $B_n(t)$ and lower bound $\sqrt{t}$



$B_5(t), b_5(t), \sqrt{t}$

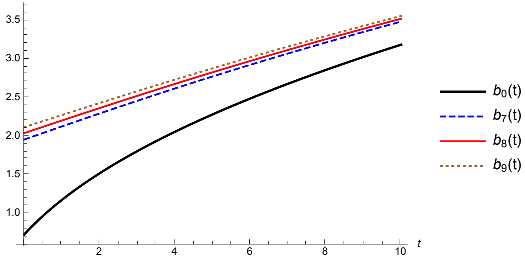


$B_{10}(t), b_{10}(t), \sqrt{t}$

Plots of $B_n(t)$ (dot-dash line), $b_n(t)$ (solid line) and $\sqrt{t}$ (dashed line) for $n = 5$ and $n = 10$.

Comparison between $b_0,\ b_7, b_8$ and b_9

Lemma. $b_n'(t) > 0$ for any $t \geq \left(\frac{(2n+3)n^3}{(2n+1)(n+1)}\right)^{2/3}$.



Relation with Painlevé II

The **second Painlevé equation P_{II}** is

$$\frac{d^2 w}{dz^2} = 2w^3 + zw + \alpha \tag{9}$$

and it has solutions which we denote by $w(z;\alpha)$.
The Backlünd transformations for P_{II} are

$$w(z;\alpha+1) = -w(z;\alpha) - \frac{2\alpha+1}{2w^2(z;\alpha) + 2w'(z;\alpha) + z}$$
$$w(z;\alpha-1) = -w(z;\alpha) - \frac{2\alpha-1}{2w^2(z;\alpha) - 2w'(z;\alpha) + z}$$

The solutions $w(z;\alpha)$ and $w(z;\alpha \pm 1)$ are related by

$$\frac{\alpha+\frac{1}{2}}{w(z;\alpha+1) + w(z;\alpha)} + \frac{\alpha-\frac{1}{2}}{w(z;\alpha) + w(z;\alpha-1)} + 2w(z;\alpha)^2 + z = 0, \tag{10}$$

With $\alpha = n + \frac{1}{2}$ and $z = -2^{1/3}t$, then $y_n(t) = -2^{1/3} w(-2^{1/3}t; n+\frac{1}{2})$ satisfies

$$\frac{n+1}{y_{n+1} + y_n} + \frac{n}{y_n + y_{n-1}} - y_n^2 + t = 0, \tag{11}$$

which corresponds to the **alternative discrete Painlevé I equation (alt d-P_I)**.

Relation with Painlevé II (cont.)

For $\alpha = n + \frac{1}{2}$ the $\mathrm{P_{II}}$ equation has solutions in terms of Airy functions
– see (DLMF, 32.10(ii) on p. 735) or (Clarkson, 2006 - §7.1 on p. 373).

One has the solution $w(z; \frac{1}{2}) = -\phi'(z)/\phi(z)$, where

$$\phi(z) = C_1 \,\mathrm{Ai}(-2^{-1/3}z) + C_2 \,\mathrm{Bi}(-2^{-1/3}z),$$

with C_1 and C_2 arbitrary constants.

Observe that $w(z; \frac{1}{2})$ only depends on the ratio C_1/C_2. We now have that

$$y_0(t) = -2^{-1/3} w(-2^{1/3}t; \tfrac{1}{2}) = 2^{-1/3} \frac{\phi'(-2^{1/3}t)}{\phi(-2^{1/3}t)}$$

and since

$$\phi(-2^{-1/3}t) = C_1 \,\mathrm{Ai}(t) + C_2 \,\mathrm{Bi}(t), \quad \phi'(-2^{-1/3}t) = -2^{1/3}\big(C_1 \,\mathrm{Ai}'(t) + C_2 \,\mathrm{Bi}'(t)\big),$$

we find

$$y_0(t) = -\frac{C_1 \,\mathrm{Ai}'(t) + C_2 \,\mathrm{Bi}'(t)}{C_1 \,\mathrm{Ai}(t) + C_2 \,\mathrm{Bi}(t)}.$$

The "Airy-type" solutions of

$$\frac{\mathrm{d}^2 y_n}{\mathrm{d}t^2} = 2y_n^3 - 2ty_n - 2n - 1, \tag{12}$$

and

$$\frac{\mathrm{d}^2 x_n}{\mathrm{d}t^2} = \frac{1}{2x_n}\left(\frac{\mathrm{d}x_n}{\mathrm{d}t}\right)^2 + 4x_n^2 + 2tx_n - \frac{n^2}{2x_n}. \tag{13}$$

respectively have the form

$$y_n(t;\theta) = \frac{\mathrm{d}}{\mathrm{d}t}\ln\frac{\tau_n(t;\theta)}{\tau_{n+1}(t;\theta)}, \tag{14}$$

$$x_n(t;\theta) = -\frac{\mathrm{d}^2}{\mathrm{d}t^2}\ln\tau_n(t;\theta), \tag{15}$$

where

$$\tau_n(t;\theta) = \det\left[\frac{\mathrm{d}^{j+k}}{\mathrm{d}t^{j+k}}\varphi(t;\theta)\right]_{j,k=0}^{n-1}, \tag{16}$$

with $\tau_0(t;\theta) = 1$, and

$$\varphi(t;\theta) = \cos(\theta)\,\mathrm{Ai}(t) + \sin(\theta)\,\mathrm{Bi}(t), \tag{17}$$

The simplest non-trivial "Airy-type" solutions of (12) and (13) respectively have the form

$$y_0(t;\theta) = -\frac{\mathrm{d}}{\mathrm{d}t}\ln\varphi(t;\theta) = -\frac{\cos(\theta)\,\mathrm{Ai}'(t)+\sin(\theta)\,\mathrm{Bi}'(t)}{\cos(\theta)\,\mathrm{Ai}(t)+\sin(\theta)\,\mathrm{Bi}(t)}, \tag{18a}$$

$$x_1(t;\theta) = -\frac{\mathrm{d}^2}{\mathrm{d}t^2}\ln\varphi(t;\theta) = \left[\frac{\cos(\theta)\,\mathrm{Ai}'(t)+\sin(\theta)\,\mathrm{Bi}'(t)}{\cos(\theta)\,\mathrm{Ai}(t)+\sin(\theta)\,\mathrm{Bi}(t)}\right]^2 - t, \tag{18b}$$

recall that $x_0(t;\theta) = 0$.

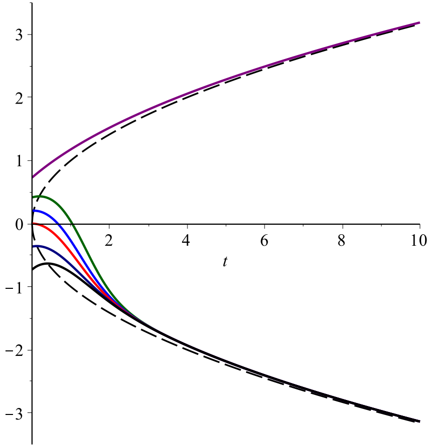
Lemma. If $y_0(t;\theta)$ is given by (18a), then

$$y_0(t;\theta) = \begin{cases} t^{1/2} + \mathcal{O}\left(t^{-1}\right), & \text{if} \quad \theta = 0, \\ -t^{1/2} + \mathcal{O}\left(t^{-1}\right), & \text{if} \quad \theta \neq 0. \end{cases} \tag{19}$$

Moreover,

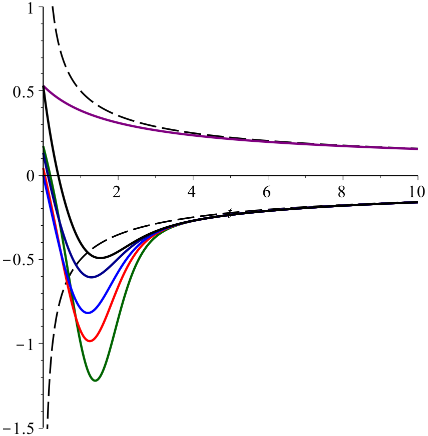
$$y_n(t;\theta) = \begin{cases} t^{1/2} + \mathcal{O}\left(t^{-1}\right), & \text{if} \quad \theta = 0, \\ -t^{1/2} + \mathcal{O}\left(t^{-1}\right), & \text{if} \quad \theta \neq 0 \end{cases}$$

Plots of $y_0(t;\theta)$



Plots of $y_0(t;\theta)$ for $\theta = 0$ (upper curve), $\theta = \frac{1}{20}\pi, \frac{1}{10}\pi, \frac{1}{6}\pi, \frac{1}{3}\pi, \frac{1}{2}\pi$ (lower curves, with $\theta = \frac{1}{20}\pi$ the highest and $\theta = \frac{1}{2}\pi$ the lowest), and the curves $y = \pm\sqrt{t}$ (dashed lines).

Plots of $x_1(t;\theta)$



Plots of $x_1(t;\theta)$ for $\theta = 0$ (upper curve), $\theta = \frac{1}{20}\pi, \frac{1}{10}\pi, \frac{1}{6}\pi, \frac{1}{3}\pi, \frac{1}{2}\pi$ (lower curves, with $\theta = \frac{1}{20}\pi$ the lowest and $\theta = \frac{1}{2}\pi$ the highest), and the curves $x = \pm 1/(2\sqrt{t})$ (dashed lines).

Lemma. If $a_n(t)$ and $b_n(t)$ satisfies the recurrence relation

$$a_n + a_{n+1} = b_n^2 - t,$$
$$a_n(b_n + b_{n-1}) = n,$$

with

$$a_0(t) = 0, \qquad b_0(t) = -\frac{\mathrm{Ai}'(t)}{\mathrm{Ai}(t)}, \tag{20}$$

where $\mathrm{Ai}(t)$ is the Airy function, then as $t \to \infty$

$$a_n(t) = \frac{n}{2\,t^{1/2}} + \mathcal{O}\left(t^{-2}\right), \qquad \text{for} \quad n \geq 1, \tag{21a}$$
$$b_n(t) = t^{1/2} + \mathcal{O}\left(t^{-1}\right), \qquad \text{for} \quad n \geq 0. \tag{21b}$$

Corollary. The positive solution $b_n(t)$ corresponds to the initial value $b_0(t) = -\mathrm{Ai}'(t)/\mathrm{Ai}(t)$.

$b_n(t)$ is an increasing function of both t and n

- (conjecture) $b_n'(t) \geq 0$
- $\lim_{t \to +\infty} \left(b_n'(t)\sqrt{t} \right) = \frac{1}{2}$
- $b_n'(t) < b_n^2(t) - t$
- $\frac{n \ t^{1/4}}{2\sqrt{n + \frac{3}{2} + t^{3/2}}} \ \leq \ a_n \ \leq \ \frac{n}{2\sqrt{t}}$

For fixed t with $t > 0$ then

$$\sqrt{t} < b_n(t) < b_{n+1}(t), \tag{22a}$$

$$\frac{1}{2\sqrt{t}} > \frac{a_n(t)}{n} > \frac{a_{n+1}(t)}{n+1} > 0. \tag{22b}$$

Remarks

The Airy solutions are used in the asymptotic analysis of the partition function and free energy in the cubic Hermitian random matrix model (in Bleher, Dea no & Yattselev'18)

In random matrix theory, the pure Ai case of PII special function solutions arises in the calculation of averages of powers of the characteristic polynomial in the GUE (Gaussian unitary ensemble)

The asymptotic behavior of these Airy solutions has been investigated recently by Deaño'18 after Clarkson'18 discussed the asymptotic results restricted to the real line, and only for even values of n in the oscillatory regime.

Some references

- P. Bleher, A. Deaño, Topological expansion in the cubic random matrix model, Int. Math. Res. Notes. (12) (2013), 2699-2755.
- P.A. Clarkson, A.F. Loureiro, W. Van Assche, Unique positive solution for alternative d-PI, J. Difference Equ. Appl., 22 (2016) 656-675
- P.A. Clarkson, On Airy solutions of the second Painlevé equation, Stud. Appl. Math. 137 (2016), 93-109.
- A. Deaño, Large z Asymptotics for Special Function Solutions of Painlevé II in the Complex Plane, SIGMA 14 (2018), 107.
- A. Magnus, Painlevé-type differential equations for the recurrence coefficients of semi-classical orthogonal polynomials, J. Comput. Appl. Math. 57 (1995), 215-237.
- A. Deaño, D. Huybrechs, A.B.J. Kuijlaars, Asymptotic zero distribution of complex orthogonal polynomials associated with Gaussian quadrature, J. Approx. Theory 162 (2010), 2202-2224.
- W. Van Assche, Galina Filipuk, Lun Zhang, Multiple orthogonal polynomials associated with an exponential cubic weight, J. Approx. Theory 190 (2015), 1-25.